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Graph family operations

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Abstract

In previous papers, Catlin introduced four functions, denoted $\mathscr{S}^O$, $\mathscr{S}^R$, $\mathscr{S}^C$, and $\mathscr{S}^H$, between sets of finite graphs. These functions proved to be very useful in establishing properties of several classes of graphs, including supereulerian graphs and graphs with nowhere zero k-flows for a fixed integer $k \geqslant 3$. Unfortunately, a subtle error caused several theorems previously published in Catlin (Discrete Math. 160 (1996) 67–80) to be incorrect. In this paper we correct those errors and further explore the relations between these functions, showing that there is a sort of duality between them and that they act as inverses of one another on certain sets of graphs.

Keywords: Graph family; Complete family; Free family

1. Introduction

When our dear friend and mentor, Paul Catlin, died, he left behind a manuscript which has evolved into this paper. Unfortunately, there was a subtle error both in his manuscript and in Paul's previously published paper [5]. It was sufficiently subtle that we, his coauthors, also failed to notice it as we initially revised the manuscript. The error resided principally in failing to notice that knowing the set of edges of a subgraph H of graph G is preserved under a contraction of G to G_0, even after erasing loops, does not assure that $H \subseteq G_0$. Stated clearly, as here, it seems trivial, but in the midst of a long and difficult proof it is easy to miss. Fortunately, one of our referees noticed a problem with connectedness, and that observation led us to recognize the key problem. We are pleased to have the opportunity here both to thank the referee and to repair

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the results in [5], one of Paul's most important papers. We also extend these results along the lines Paul had mapped out.

Throughout this paper, we confine ourselves to the universe $\mathscr{U}$ of all finite graphs whose vertex sets are subsets of $\mathbf{N}$, the set of all nonnegative integers, and whose edges are integers mapped into unordered pairs of those vertices. We generally follow the terminology of Bondy and Murty [1], except that a *graph* has no loops, all graphs in this paper are finite, and $\boldsymbol{nG}$ stands for a vertex-disjoint union of n copies of graph G. We use $\boldsymbol{n * K_2}$ for any graph with two vertices joined by n parallel edges. A graph G is *edgeless* if $|E(G)| = 0$, but we briefly describe any graph with at least one edge as being *nontrivial*. A group A is *trivial* if $|A| = 1$. We use $\mathbf{Z}^+$ for the set of positive integers. If H_1 and H_2 are two subgraphs of a graph G, then $H_1 \cup H_2$ denotes the subgraph of G with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. If nothing is said about the relationship between $V(H_1)$ and $V(H_2)$ (e.g., they are not described as subgraphs of a given graph), then $H_1 \cup H_2$ denotes the disjoint union of H_1 and H_2.

A *contraction* of G is a graph G' obtained from G by contracting a set (possibly empty) of edges and deleting any loops generated in the process. If G' is a contraction of G, then we say that G is *contractible* to G'. A contraction of a subgraph of G is called a *minor* of G. When H is a subgraph of G, then the contraction of G obtained by contracting the edges in H and deleting resulting loops is denoted G/H. Note that each component of H becomes a vertex of G/H. For graphs G and H, by $H \leqslant G$ we mean that H is a minor of G; by $H \subseteq G$ we mean that H is a subgraph of G, and by $H \cong G$ we mean that H is isomorphic to G. We use the term *family of graphs* or *graph family* to refer to any subset of $\mathscr{U}$ which is closed under isomorphism. A graph family $\mathscr{S}$ is *closed under contraction* if $G \in \mathscr{S}$ and H a contraction of G together imply that $H \in \mathscr{S}$. A graph family $\mathscr{S}$ is *closed under taking subgraphs* if $G \in \mathscr{S}$ and $H \subseteq G$ together imply that $H \in \mathscr{S}$.

A family $\mathscr{S}$ of graphs is a *lower ideal* if $\mathscr{S}$ is closed under minors (i.e. $G \in \mathscr{S}$ and $H \leqslant G$ together imply $H \in \mathscr{S}$). A family $\mathscr{S}$ of graphs is a *contraction lower ideal* if $\mathscr{S}$ is closed under contraction (i.e., $G \in \mathscr{S}$ and $H \subseteq G$ together imply $G/H \in \mathscr{S}$), and $\mathscr{S}$ is a *subgraph lower ideal* if $\mathscr{S}$ is closed under taking subgraphs (i.e., $G \in \mathscr{S}$ and $H \subseteq G$ together imply $H \in \mathscr{S}$). Robertson and Seymour have found that lower ideals have nicely describable internal and external structures, which led to many wonderful discoveries. Here we examine separately and then together the two concepts of subgraph lower ideals and contraction lower ideals.

Examples of graph families that are contraction lower ideals but not lower ideals are the family of supereulerian graphs and the family of graphs with nowhere zero k-flows, both of which will be introduced below. For literature on these two topics, see [4,8].

Let G be a graph and let $O(G)$ denote the set of vertices of odd degree in G. Then G is *even* if $O(G) = \emptyset$, and G is *eulerian* if G is both even and connected. A graph G is *supereulerian* if G has a spanning eulerian subgraph. Following Catlin [4], the family of supereulerian graphs is denoted by $\mathscr{SL}$.

Let G be a digraph. For a vertex $v \in V(G)$, let

$$E_G^-(v) = \{(u,v) \in E(G) : u \in V(G)\} \quad \text{and} \quad E_G^+(v) = \{(v,u) \in E(G) : u \in V(G)\}.$$

The subscript G may be omitted when G is understood from the context. Let $E(v) = E^+(v) \cup E^-(v)$.

Let A be a nontrivial additive abelian group and let A^* denote the set of nonzero elements in A. For a digraph G, define $F(G,A)$ to be the set of all functions from $E(G)$ into A, and $F^*(G,A)$ to be the set of all functions from $E(G)$ into A^*. For each $f \in F(G,A)$, the *boundary* of f is a function $\partial f : V(G) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where '$\sum$' refers to the addition in A and an empty sum has value zero.

Let S be a nonempty set and let A be a group. Throughout this paper, we shall adopt the following convention: if $X \subseteq S$ and if $f : X \to A$ is a function, then we regard f as a function $f : S \to A$ such that $f(e) = 0$ for all $e \in S - X$.

Let G be an undirected graph and A be an abelian group. Define $Z(G,A)$ to be the set of all functions $b : V(G) \to A$ such that $\sum_{v \in V(G)} b(v) = 0$. A graph G is *A-connected* if G has an orientation G' such that for every function $b \in Z(G,A)$, there is a function $f \in F^*(G',A)$ such that $b = \partial f$. In particular, K_1 is A-connected for any abelian group A. It is observed in [9] that whether G is A-connected is independent of the orientation of G. Let $[A]$ denote the family of all A-connected graphs.

An *A-nowhere-zero-flow* (abbreviated as A-NZF) in G is a function $f \in F^*(G,A)$ such that $\partial f \cong 0$. The nowhere-zero-flow problems were introduced by Tutte [16], and were surveyed by Jaeger [8]. Tutte [16] showed that if A_1 and A_2 are two abelian groups with $|A_1| = |A_2|$, then a graph G has an A_1-NZF if and only if it has an A_2-NZF. Thus, an A-NZF is also called a k-**NZF**, where $k = |A|$. Following Jaeger [8], let F_k denote the family of graphs that have k-NZFs.

Jaeger et al. [9], generalized the concept of A-NZF to A-connectivity, or group connectivity. A concept similar to group connectivity was independently introduced in [10], with a different motivation from [9].

It is immediate that a nontrivial graph G is in $\mathcal{SL}$ or in F_k $(k \geqslant 2)$ only if G is 2-edge-connected. Therefore both $\mathcal{SL}$ and F_k are not lower ideals. On the other hand, it also follows from the definitions that both $\mathcal{SL}$ and F_k are contraction lower ideals.

Contraction lower ideals are interesting due to a reduction method first introduced by Catlin [2]. Some prior work on certain special contraction lower ideals (called 'complete families' by Catlin) can be found in [5,13]. In this paper, we continue the work, including correcting errors found in [5]. When we give new statements or new proofs to theorems of [5], we use the style 'Theorem w.x ([5] y.z'), where 'w.x' is the number of the theorem in this paper and 'y.z' is the number of the theorem in [5].

2. Preliminaries

An *elementary contract-homomorphism* (or an *elementary CH-morphism* for short) of a graph G is a transformation from G to a graph G' obtained from G by identifying two vertices lying in the same component in G and by deleting any loops that might result. A *CH-morphism* of G is a transformation from G to a graph G' obtained from G by a sequence of elementary CH-morphisms. We call G' a *CH-morph* of G. As the sequence of elementary CH-morphisms may be an empty sequence, any graph G is a CH-morph of itself. A graph family $\mathcal{S}$ is *closed under CH-morphisms* if $G \in \mathcal{S}$ and H a CH-morph of G together imply that $H \in \mathcal{S}$.

The first three operations below were defined in [5], and the fourth one in [3]. Let $\mathcal{U}'$ be the set of all families of finite graphs in $\mathcal{U}$. Define four functions from $\mathcal{U}'$ to $\mathcal{U}'$ as follows:

$$\begin{aligned}
\mathcal{S}^O &= \{H : \text{for every graph } G \text{ with } H \subseteq G, G \in \mathcal{S} \Leftrightarrow G/H \in \mathcal{S}\},\\
\mathcal{S}^R &= \{G : G \text{ has no nontrivial subgraph in } \mathcal{S}\},\\
\mathcal{S}^C &= \{G : G \text{ has no nontrivial contraction in } \mathcal{S}\},\\
\mathcal{S}^H &= \{G : G \text{ has no nontrivial CH-morph in } \mathcal{S}\}.
\end{aligned}$$

The graph family $\mathcal{S}^O$ is called the *kernel* of $\mathcal{S}$ in [5].

Example 2.1. Let $\mathcal{S} = \{K_3\}$. Then $\mathcal{S}^O = \{K_1, 2K_1, 3K_1\} \cup \{G : \omega(G) \geqslant 4\}$, $\mathcal{S}^R$ is the set of all K_3-free graphs, $\mathcal{S}^C =$ the set of all graphs G such that G is not connected or G is connected and does not have a set C of 3 edges such that $G - C$ is the vertex-disjoint union of three connected graphs H_1, H_2, and H_3 such that the edges of C can be labeled e_1, e_2, e_3 so that e_i joins a vertex of H_i with a vertex of H_{i+1} for each i, taking $3 + 1 = 1$, and $\mathcal{S}^H =$ the set of all graphs G such that G is not connected or G is connected and does not have a set C of 3 edges such that $G - C$ is the vertex-disjoint union of three graphs H_1, H_2, and H_3 such that the edges of C can be labeled e_1, e_2, e_3 so that e_i joins a vertex of H_i with a vertex of H_{i+1} for each i, taking $3 + 1 = 1$.

Proof. Suppose $H = nK_1$ for some $n \in \{1, 2, 3\}$ and suppose $H \subseteq G$ for graph G. If $G \in \mathcal{S}$, then $G/H = G \in \mathcal{S}$. Moreover, if $G/H \in \mathcal{S}$, then $G = G/H \in \mathcal{S}$. Thus $nK_1 \in \mathcal{S}^O$ for $n \in \{1, 2, 3\}$. On the other hand, suppose H has the property that, for every graph G with $H \subseteq G$, $G \in \mathcal{S} \Leftrightarrow G/H \in \mathcal{S}$. Then, for such H and G, $3 = |E(G)| \geqslant |E(G/H)| + |E(H)| \geqslant |E(G/H)| = 3$, so $H = nK_1$ for some $n \in \{1, 2, 3\}$. But if $\omega(H) \geqslant 4$, then both G and G/H must have at least four vertices and so cannot be in $\mathcal{S}$. Thus the condition is satisfied vacuously in this case.

The description of $\mathcal{S}^R$ is immediate from the definitions. For $\mathcal{S}^C$ and $\mathcal{S}^H$, if G is not connected, then every CH-morph (including contractions) of G is not connected, and so $G \in \mathcal{S}^C \cap \mathcal{S}^H$. Suppose G is connected and does not have a set C of 3 edges such that $G - C$ is the vertex-disjoint union of three graphs H_1, H_2, and H_3 such that the edges of C can be labeled e_1, e_2, e_3 so that e_i joins a vertex of H_i with a vertex of H_{i+1} for each i, taking $3+1=1$. (Note: The graphs H_i are not necessarily connected.) Suppose further that α is a CH-morphism from G to K_3. Let the vertices of $\alpha(G)$ be v_1, v_2, and v_3, and let $\alpha^{-1}(v_i) = H_i$ for each i. Then $V(H_i) \cap V(H_j) = \emptyset$ for each distinct pair i and j. Since any edge from H_i to H_j becomes an edge from v_i to v_j under α, and since there is only one such edge in $\alpha(G)$ for each distinct pair i and j, the three edges of G which become the three edges of $\alpha(G)$ constitute a set C of edges such that $G - C$ has the three subgraphs H_i as described, contrary to assumption. Thus $\mathscr{S}^H$ contains the claimed graphs. But if G is connected and does have a set C of 3 edges such that $G - C$ has the described decomposition into three subgraphs, then (recalling G is connected, so the graphs H_i can each be contracted to a single vertex) contracting those three subgraphs to three distinct vertices yields a K_3, and so $G \notin \mathscr{S}^H$.

The proof of the description of $\mathscr{S}^C$ is similar, except that each of the subgraphs H_i is connected. □

Example 2.2. Let $\mathscr{S}$ be the family of all forests, then $\mathscr{S}^O = \mathscr{S}^R = \{nK_1 : n \in \mathbf{Z}^+\}$, and $\mathscr{S}^C$ and $\mathscr{S}^H$ are both the family of all graphs such that every nontrivial component is 2-edge-connected.

Proof. Trivially, if $H = nK_1$ for some integer n, then $G/H = G$ for every graph G having H as a subgraph, so $\{nK_1 : n \in \mathbf{Z}^+\} \subseteq \mathscr{S}^O$. Suppose H has the property that $G \in \mathscr{S} \Leftrightarrow G/H \in \mathscr{S}$ for every graph G containing H, and suppose H has an edge e. Form graph G from H by adding a second edge parallel to e in H. Then G/H is nK_1 for some integer n, and so it is in $\mathscr{S}$. But G has a digon and so cannot be in $\mathscr{S}$. This contradiction shows our claim for $\mathscr{S}^O$.

Note that $\{nK_1 : n \in \mathbf{Z}^+\} \subseteq \mathscr{S}^R$ by definition. Suppose H has an edge e. Then H has K_2 as a subgraph, and so H is not in $\mathscr{S}^R$. This establishes the claim for $\mathscr{S}^R$.

Since a contraction or CH-morph of a 2-edge-connected graph is 2-edge-connected or edgeless, the family of all graphs such that every nontrivial component is 2-edge-connected is contained in $\mathscr{S}^C \cap \mathscr{S}^H$. But suppose G is a graph having a nontrivial component H which is not 2-edge-connected. Then there is at least one cut-edge in H. Contract H to that cut-edge, and contract all other components of G to vertices. The result is a nontrivial forest, and so $G \notin \mathscr{S}^C \cup \mathscr{S}^H$. □

Example 2.3. Let $\mathscr{S} = \{K_1\} \cup \{\text{2-edge-connected graphs}\}$. Then $\mathscr{S}^O$ is the the family of all graphs such that every nontrivial component is 2-edge-connected, $\mathscr{S}^R$ is the family of forests, $\mathscr{S}^C = \{G : G \text{ is a tree}\} \cup \{G : G \text{ is disconnected}\}$, and $\mathscr{S}^H = \{K_1, K_2\} \cup \{G : G \text{ is disconnected}\}$.

Proof. Suppose H is a graph each of whose components is either K_1 or 2-edge-connected. Let G be a graph containing H as a subgraph. If $G \in \mathscr{S}$, then contracting the 2-edge-connected components of H leaves a 2-edge-connected graph or K_1, so $G/H \in \mathscr{S}$. If $G/H \in \mathscr{S}$, then certainly G is connected. If G has an edge e whose removal disconnects G, we notice that the edge is not in G/H, since G/H is either K_1 or is 2-edge-connected. But every nontrivial component of H is 2-edge-connected also, so e cannot be in a component of H either. Thus G is 2-edge-connected. Conversely, suppose H has the property that $G \in \mathscr{S} \Leftrightarrow G/H \in \mathscr{S}$ for every graph G containing H, and suppose some component H_0 of H has a cut edge e. Let e have ends v_0 and w, and let the other components of H be $H_1, H_2, \ldots, H_c$, $c \geqslant 0$. Let H_i include a vertex v_i for each $i \in \{1, 2, \ldots, c\}$. If $c > 0$, form G by adding to H an edge from v_i to v_{i+1} for each $i \in \{0, 1, \ldots, c-1\}$ and an edge from v_c to v_0. If $c = 0$, form G by identifying v_0 with one vertex of K_3. Then G/H is a cycle and so is in $\mathscr{S}$, but e is a cut edge of G, contrary to our assumption. Hence no component of H has a cut-edge and the first claim is proved.

Let $G \in \mathscr{S}^R$. Then G has no nontrivial subgraph in $\mathscr{S}$. If G has a cycle C, then $C \in \mathscr{S}$, a contradiction. Thus G has no cycle and so is a forest. But if G is a forest, it has no nontrivial subgraph in $\mathscr{S}$. Thus $\mathscr{S}^R$ is the family of all forests.

If G is disconnected, then every contraction and every CH-morph of G has at least two components and so is not in $\mathscr{S}$. Thus $G \in \mathscr{S}^H \cap \mathscr{S}^C$ in this case. Continuing with $\mathscr{S}^C$ alone, if G is a tree, then every contraction of G is a tree and so is not in $\mathscr{S}$. Thus the graphs claimed for $\mathscr{S}^C$ are in that graph family. Going on to $\mathscr{S}^H$, if G is K_1 or K_2, then G is not 2-edge-connected. Moreover, neither one has any proper nontrivial CH-morph. Thus both are in $\mathscr{S}^H$. It follows that $\mathscr{S}^H$ includes all of the claimed graphs. But if G is connected and is neither K_1 nor K_2, then it is either $n * K_2$ with $n \geqslant 2$, and so is in $\mathscr{S}$, or it has a path of length at least two and so has a CH-morph to $n * K_2 \in \mathscr{S}$. Thus $\mathscr{S}^H$ includes exactly the claimed graphs. Moreover, if connected graph G has a cycle, then contracting the remaining vertices of G to vertices of that cycle produces a two-edge-connected graph, and so $G \notin \mathscr{S}^C$. Thus $\mathscr{S}^C$ includes exactly the claimed graphs. $\square$

It is interesting to note that $\mathscr{S}^0$ of Example 2.3 is the same as $\mathscr{S}^C = \mathscr{S}^H$ of Example 2.2. Next, we look at some elementary properties of these graph operations.

Lemma 2.4. *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be arbitrary graph families.* If $\mathscr{S}_1 \subseteq \mathscr{S}_2$ *then*

(a) $\mathscr{S}_2^R \subseteq \mathscr{S}_1^R$;
(b) $\mathscr{S}_2^C \subseteq \mathscr{S}_1^C$; *and*
(c) $\mathscr{S}_2^H \subseteq \mathscr{S}_1^H$.

Proof. Suppose $\mathscr{S}_1 \subseteq \mathscr{S}_2$.

(a) Let $G \in (\mathscr{S}_2)^R$. Then G has no nontrivial subgraph in $\mathscr{S}_2$, and by $\mathscr{S}_1 \subseteq \mathscr{S}_2$ no nontrivial subgraph in $\mathscr{S}_1$. Therefore, $G \in \mathscr{S}_1^R$.

(b) Pick $G \in (\mathscr{S}_2)^C$. Then no nontrivial contraction of G lies in $\mathscr{S}_2$, and by $\mathscr{S}_1 \subseteq \mathscr{S}_2$ no nontrivial contraction of G lies in $\mathscr{S}_1$. Therefore, $G \in \mathscr{S}_1^C$.

(c) Imitate the proof of (b). $\square$

Catlin [5] showed that, in certain circumstances, the converses of Lemma 2.4(a) and (b) are also true. In Section 7 we will state his result precisely and extend it to conditions for which the converse of Lemma 2.4(c) is true.

Lemma 2.5 (Catlin [5, 4.3]). *For any graph family $\mathcal{S}$, $\mathcal{S}^R$ is closed under taking subgraphs.*

Proof. To see that $\mathcal{S}^R$ is closed under taking subgraphs, let $G \in \mathcal{S}^R$ and let $H \subseteq G$. If $H \notin \mathcal{S}^R$, then H has a nontrivial subgraph in $\mathcal{S}$. But any subgraph of H is a subgraph of G, so G has a nontrivial subgraph in $\mathcal{S}$. Hence $G \notin \mathcal{S}^R$. This contradiction establishes the lemma. □

Lemma 2.6 (Catlin [5, 4.8]). *For any graph family $\mathcal{S}$, $\mathcal{S}^C$ is closed under contractions.*

Lemma 2.7. *For any graph family $\mathcal{S}$, $\mathcal{S}^H$ is closed under CH-morphisms.*

The proofs of Lemmas 2.6 and 2.7 parallel the proof of Lemma 2.5, using the transitivity of the contraction and CH-morphism operations.

Lemma 2.8. *Let $\mathcal{S}$ be a graph family. Then each of the following holds*:

(a) $\{nK_1 : n \in \mathbf{Z}^+\} \subseteq \mathcal{S}^R$ *and* $\mathcal{S} \cap \mathcal{S}^R \subseteq \{nK_1 : n \in \mathbf{Z}^+\}$,
(b) $\{nK_1 : n \in \mathbf{Z}^+\} \subseteq \mathcal{S}^C$ *and* $\mathcal{S} \cap \mathcal{S}^C \subseteq \{nK_1 : n \in \mathbf{Z}^+\}$, *and*
(c) $\{nK_1 : n \in \mathbf{Z}^+\} \subseteq \mathcal{S}^H$ *and* $\mathcal{S} \cap \mathcal{S}^H \subseteq \{nK_1 : n \in \mathbf{Z}^+\}$.

Proof. The first part of each of these is immediate from the definitions. Let $G \in \mathcal{S} \cap \mathcal{S}^R$. If G has a subgraph H with $E(H) \neq \emptyset$, then G has a nontrivial subgraph in $\mathcal{S}$, contrary to the assumption that $G \in \mathcal{S}^R$, Hence $E(G) = \emptyset$, and so (a) obtains.

Let $G \in \mathcal{S} \cap \mathcal{S}^C$. If $E(G) \neq \emptyset$, then since $G = G/K_1$ is a contraction of G, and since $G \in \mathcal{S}$, G has a nontrivial contraction in $\mathcal{S}$, and so $G \notin \mathcal{S}^C$, a contradiction. Therefore, $E(G) = \emptyset$, and so (b) obtains.

Part (c) can be proved with an argument similar to that for Part (b), as any graph G is a CH-morph of G itself. □

Note that Lemmas 2.8(a) and 2.8(b), respectively, are Lemmas 4.4 and 4.5 of [5].

Lemma 2.9. *For any graph family $\mathcal{S}$,*

$$\mathcal{S}^H \subseteq \mathcal{S}^C.$$

Proof. Suppose $G \in \mathcal{S}^H$. Then there is no nontrivial CH-morph of G in $\mathcal{S}$, and so there is no nontrivial contraction of G in $\mathcal{S}$. Therefore, $G \in \mathcal{S}^C$ and hence $\mathcal{S}^H \subseteq \mathcal{S}^C$. □

3. Complete graph families

In our studies of these four graph family operations, two special kinds of graph families, the 'complete' and the 'free' graph families, have been particularly significant. We discuss 'complete' graph families in this section and 'free' graph families in the next section.

Define a family $\mathscr{C}$ of graphs to be *complete* if $\mathscr{C}$ satisfies these three axioms:

(C1) $\mathscr{C}$ contains all edgeless graphs;
(C2) $\mathscr{C}$ is closed under contraction;
(C3) $H \subseteq G, H \in \mathscr{C}$, $G/H \in \mathscr{C}$ then $G \in \mathscr{C}$.

Notice that complete graph families are contraction lower ideals by (C2).

In [5], Catlin proved this characterization of kernels:

Theorem 3.1 (Catlin [5, 3.4]). *For any graph family $\mathscr{S}$, closed under contraction, these are equivalent*:

(a) *$\mathscr{S}$ is a complete graph family*;
(b) $\mathscr{S} = \mathscr{S}^O$;
(c) *$\mathscr{S}$ is the kernel of some contraction lower ideal.*

Example 3.2. For an integer $k \in \mathbf{N}$, let $\mathscr{C}_{\omega > k}$ be the graph family

$$\mathscr{C}_{\omega > k} = \{nK_1 : n \in \mathbf{Z}^+\} \cup \{G : \omega(G) > k\},$$

where $\omega(G)$ is the number of components of graph G. Because contractions do not change the number of components of the graphs being contracted, it is easy to show that $\mathscr{C}_{\omega > k}$ is a complete graph family for each integer k. Since $\mathscr{C}_{\omega > k}$ is complete, by Theorem 3.1, $\mathscr{C}_{\omega > k} = (\mathscr{C}_{\omega > k})^O$. We restrict our attention to the case $k = 1$.

Theorem 3.3. (a) $(\mathscr{C}_{\omega > 1})^R = \{nK_1 : n \in \mathbf{Z}^+\} \cup \{n * K_2 : n \in \mathbf{Z}^+\}$.
(b) $(\mathscr{C}_{\omega > 1})^C = \{nK_1 : n \in \mathbf{Z}^+\} \cup \{G : \omega(G) = 1\}$.
(c) $(\mathscr{C}_{\omega > 1})^H = \{nK_1 : n \in \mathbf{Z}^+\} \cup \{G : \omega(G) = 1\}$.

Proof. For (a), suppose G has three or more vertices and at least one edge. Then, by erasing edges if necessary, we can find a nontrivial subgraph H of G with at least two components; necessarily, $H \in \mathscr{C}_{\omega > 1}$. Thus $G \notin (\mathscr{C}_{\omega > 1})^R$. But if G has only one or two vertices, then it is in one of the two sets claimed in (a).

For (b) and (c), the results are immediate from the fact that contractions and CH-morphisms do not change the number of components of a graph. □

In [5], the author stated the theorem that, if $\mathscr{C}$ is a complete graph family, and if H is a graph containing subgraph H_1 and H_2 such that $H = H_1 \cup H_2$, then $H \in \mathscr{C}$ if $H_1, H_2 \in \mathscr{C}$. But this is false if $\mathscr{C} = \mathscr{C}_{\omega > 1}$, for consider the 4-cycle H having the vertices a, b, c, and d in order around the cycle. Then H_1 having vertex set $V(H)$

and edges ab and cd and H_2 having vertex set $V(H)$ and edges bc and da are both disconnected and so are in $\mathscr{C}_{\omega>1}$. But $H = H_1 \cup H_2$, and, having only one component, is not in $\mathscr{C}_{\omega>1}$.

A careful reading of the proof in [5] shows that the author assumed that every component of a graph in a complete graph family was also in the complete graph family. But even with this assumption, the published proof is faulty. We remedy these problems with the following definition and two theorems. The first of these theorems is a strengthening of Theorem 3.5 of [5].

Theorem 3.4 ([5] 3.5). *Any complete graph family is closed under edge-addition between vertices of the same component.*

Proof. Let $\mathscr{C}$ be a complete graph family, let $G \in \mathscr{C}$, and let e be an edge not in G joining distinct vertices of the same component of G. Since $G \in \mathscr{C}$, by (a) $\Rightarrow$ (b) of Theorem 3.1, $G \in \mathscr{C}^O$, and so $G + e \in \mathscr{C}(G + e)/G \in \mathscr{C}$. But $(G + e)/G = nK_1$ for some integer n because e joins vertices of a single component of G and so is reduced to a loop and erased in $(G + e)/G$. Thus, $G + e \in \mathscr{C}$. $\square$

Corollary 3.5. *Every complete graph family is closed under CH-morphisms.*

Proof. Consider the elementary CH-morphism α which identifies two vertices x and y of the same component of a graph G. If the graph is in a complete graph family $\mathscr{C}$, by Theorem 3.4 we may add an edge e joining x and y to form G', and $G' \in \mathscr{C}$. But then contracting edge e in G' has the same effect as applying α to G and leaves the graph in $\mathscr{C}$ by (C2). $\square$

Definition. A graph family $\mathscr{S}$ is *stippled* if the implication $G \in \mathscr{S} \rightarrow G \cup nK_1 \in \mathscr{S}$ is always true. ('To stipple' is 'to draw with dots'.)

In [5], Catlin proved

Lemma 3.6 (Catlin [5, 3.6]). *If $\mathscr{C}$ is complete and $G \in \mathscr{C}$, then $G \cup K_1 \in \mathscr{C}$.*
In other words, every complete family is stippled.

In the following theorem, the requirement that each component of H_2 shares vertices with at most one component of H_1 does not preclude one component of H_1 sharing vertices with more than one component of H_2, nor does it preclude there being a different component of H_1 for each different component of H_2 that has such an intersection.

Theorem 3.7 ([5] 3.7). *Let $\mathscr{C}$ be a complete family of graphs. Let H_1 and H_2 be in $\mathscr{C}$ and suppose each component of H_2 shares vertices with at most one component of H_1. Then $H_1 \cup H_2 \in \mathscr{C}$.*

Proof. Consider all the pairs (C_i, D_j) of components of H_1 and H_2, respectively, such that $V_{ij} = V(C_i) \cap V(D_j) \neq \emptyset$. For all such pairs, add edges to $H_1[V_{ij}]$ to form complete graphs on those vertex sets, and let H_1' be H_1 together with these added edges. Since each component D_j of H_2 meets at most one component C_i of H_1, the resulting graph $H_1' \in \mathscr{C}$ by Theorem 3.4. Now, for every such V_{ij}, contract $H_1'[V_{ij}]$ to a vertex, and call the resulting graph H_1''. Note that $H_1'' \in \mathscr{C}$. Since each component of H_2 meets at most one component of H_1, $H_1'' \subseteq (H_1 \cup H_2)/H_2$, and if $H_1'' \neq (H_1 \cup H_2)/H_2$, then there are some number k of isolated vertices K_1 such that $H_1'' \cup kK_1 = (H_1 \cup H_2)/H_2$. But by Lemma 3.6, $H_1'' \cup kK_1 \in \mathscr{C}$. Hence $(H_1 \cup H_2)/H_2 \in \mathscr{C}$. Since also $H_2 \in \mathscr{C}$, we have $H_1 \cup H_2 \in \mathscr{C}$ by (C3). □

Definition. A *connected* graph family is a graph family $\mathscr{S}$ such that, for each graph $G \in \mathscr{S}$, every component of G is in $\mathscr{S}$.

Note that $\mathscr{S}^R$ is connected for any graph family $\mathscr{S}$. The graph family $\mathscr{C}_{\omega>1}$ shows that completeness does not assure connectedness of a graph family. However,

Theorem 3.8. *Let $\mathscr{S}$ be a stippled graph family. Then $\mathscr{S}^C$ and $\mathscr{S}^H$ are connected.*

Proof. For $\mathscr{S}^H$, let $G \in \mathscr{S}^H$ and let H_1 be a component of G. By definition, G has no nontrivial CH-morphs in $\mathscr{S}$. Suppose H_1 does have a nontrivial CH-morph $H_0 \in \mathscr{S}$. Let α be the CH-morphism mapping H_1 onto H_0, and let α' be the extension of α to G defined by letting the restriction of α' to H_1 be α and the restriction of α' to every other component of G be a mapping of the component to one of its vertices. Then α' is a CH-morphism and it maps G onto a nontrivial CH-morph of G in $\mathscr{S}$, contrary to the choice of G. This shows that $H_1 \in \mathscr{S}^H$, so $\mathscr{S}^H$ is connected.

A parallel proof shows that $\mathscr{S}^C$ is also connected. □

This theorem can fail if $\mathscr{S}$ is not stippled. For example, if $\mathscr{S} = \{G : \omega(G) = 1\}$, then $\mathscr{S}^C$ and $\mathscr{S}^H$ are both $\{K_1\} \cup \{G : \omega(G) > 1\}$, and this family is not connected.

Theorem 3.9. *Let $\mathscr{S}$ be a stippled family of graphs. If $\mathscr{S}$ is closed under taking subgraphs, then $\mathscr{S}^H$ is connected and complete.*

Proof. Let $\mathscr{S}$ be a stippled graph family which is closed under taking subgraphs. By Theorem 3.8, $\mathscr{S}^H$ is connected, and by Lemmas 2.8(c) and 2.7, $\mathscr{S}^H$ satisfies (C1) and (C2). For (C3), let $H \subseteq G$ and suppose

$$H \in \mathscr{S}^H \quad \text{and} \quad G/H \in \mathscr{S}^H. \tag{1}$$

Let H be the union of components $H_1, H_2, \ldots, H_r$. Since $\mathscr{S}^H$ is connected, $H_i \in \mathscr{S}^H$ for each i. Let $k \in \{1, 2, \ldots, r\}$. Since $H \subseteq G$, $H_1 \cup \cdots \cup H_k \subseteq G$. Define G_k to be the graph $G/(H_1 \cup \cdots \cup H_k)$ for each value of k; we take $G/(H_1 \cup \cdots \cup H_{k-1})$ to be G if $k = 1$. Then we will show that $G_{k-1} \in \mathscr{S}^H$ if $G_k \in \mathscr{S}^H$. So suppose $G_k \in \mathscr{S}^H$ but

$G_{k-1} \notin \mathscr{S}^H$. Then there is a CH-morphism α_{k-1} such that the graph $G'_{k-1} = \alpha_{k-1}(G_{k-1})$ is nontrivial and in $\mathscr{S}$. Since H_k is connected, the restriction β_{k-1} of α_{k-1} to H_k is a CH-morphism. Let $H'_k = \beta_{k-1}(H_k)$. Since the three conditions $\mathscr{S}$ closed under taking subgraphs, $H'_k \subseteq G'_{k-1}$, and $G'_{k-1} \in \mathscr{S}$ are satisfied, we have $H'_k \in \mathscr{S}$. But $H'_k \in \mathscr{S}^H$, so $H'_k = K_1$ by Lemma 2.8(c). Now extend α_{k-1} to a CH-morphism α'_{k-1} of G which first contracts every edge of $H_1 \cup \cdots \cup H_{k-1}$ and then performs α_{k-1}. (Note $\alpha'_0 = \alpha_0$.) Since α'_{k-1} can be seen as a sequence of elementary CH-morphisms, rearrange these elementary CH-morphisms to first contract all of the edges of $H_1 \cup \cdots \cup H_{k-1}$, then the edges of H_k, and finally the remaining elementary CH-morphisms of α_{k-1}; call this last subsequence γ_{k-1}. Then $\gamma_{k-1}(G_k) = \alpha'_{k-1}(G) = \alpha_{k-1}(G_{k-1}) = G_{k-1}$, a nontrivial member of $\mathscr{S}$. But $G_k \in \mathscr{S}^H$, so $\gamma_{k-1}(G_k)$ cannot be a nontrivial member of $\mathscr{S}$. Thus $G_{k-1} \in \mathscr{S}^H$.

Starting with $k = r$, repeat this argument r times. It follows that $G_0 = G \in \mathscr{S}^H$. Thus (C3) holds for $\mathscr{S}^H$ and $\mathscr{S}^H$ is complete. □

Theorem 3.10 ([5] 3.8). *Let $\mathscr{C}$ be a connected complete graph family and let G be a graph. Let E' be the set of edges of G that lie in no subgraph of G in $\mathscr{C}$. If E'' is an arbitrary minimal edge set such that $G - E''$ is in $\mathscr{C}$, then $E'' = E'$ and the maximal subgraph of G in $\mathscr{C}$ is unique.*

Proof. If $e \in E(G) - E''$ then $e \notin E'$, and so $E' \subseteq E''$. Since $\mathscr{C}$ is connected, for each edge $e \notin E'$ we may let H_e be a connected subgraph of G in $\mathscr{C}$ containing e. Letting $E(G) - E' = \{e_1, e_2, \ldots, e_n\}$, the graph $(\ldots(H_{e_1} \cup H_{e_2}) \cup \cdots \cup H_{e_{n-1}}) \cup H_{e_n}$ is in $\mathscr{C}$ by Theorem 3.7. Thus $E'' \subseteq E'$. Combining our two results, $E'' = E'$ and E'' is uniquely determined. Since the maximal connected subgraphs of G in $\mathscr{C}$ are the components of $G - E''$, they are uniquely determined. □

Example 3.11. If the complete graph family $\mathscr{C}$ in Theorem 3.10 is not connected, the result can fail. For example, let $\mathscr{C} = \mathscr{C}_{\omega>1}$ of Example 3.2, and let G be the graph with four vertices a, b, c, d and six edges distributed as follows: Two edges join a and b, two edges join c and d, one edge joins a and d, and one edge joins b and c. Let G_1 be the subgraph of G with exactly the edges ad and bc, and let G_2 be the subgraph of G having exactly the other four edges of G. Then each of G_1 and G_2 is in $\mathscr{C}$ and $G = G_1 \cup G_2$. Hence $E' = \emptyset$ in Theorem 3.10. But $G \notin \mathscr{C}$ and the minimum edge set E'' such that $G - E'' \in \mathscr{C}$ is $E'' = \{ad, bc\} \neq E'$.

The next theorem appears as Lemma 3.9 in [5] without requiring connectedness of the complete graph family $\mathscr{C}$. Although the first two paragraphs of the published proof are sound, they are reproduced here for completeness.

Theorem 3.12 ([5] 3.9). *Let $\mathscr{C}$ be a connected complete graph family, let G be a graph, and let $H \in \mathscr{C}$ be a connected subgraph of G. Let E' and E'' be as in Theorem* 3.10. *Let E^* be the set of edges of G/H that lie in no subgraph of G/H in $\mathscr{C}$. If E^{**}*

is an arbitrary minimal edge set such that $G/H - E^{**}$ *is in* $\mathscr{C}$, *then*

$$E'' = E' = E^* = E^{**}. \tag{2}$$

Proof. The first and last equalities of (2) are instances of Theorem 3.10. It remains to prove $E' = E^*$.

Let H be a connected subgraph of G with $H \in \mathscr{C}$, let $e \in E'$, and suppose $e \notin E^*$. Then e is in a subgraph H'' of G/H with $H'' \in \mathscr{C}$. Denote by G'' the subgraph of G induced by $E(H) \cup E(H'')$. Thus,

$$H \subseteq G'', \quad H \in \mathscr{C}, \quad G''/H = H'' \in \mathscr{C},$$

and so by (C3), $G'' \in \mathscr{C}$. But then $e \in E(H'') \subseteq E(G'')$, contrary to $e \in E'$. Therefore $E' \subseteq E^*$.

Let $e \in E^* - E'$. Then $e \notin E'$, so, by Theorem 3.10, G has a unique maximal subgraph $H_0 \in \mathscr{C}$ such that $e \in E(H_0)$. Since $\mathscr{C}$ is connected, the component H_0' of H_0 containing e is in $\mathscr{C}$. Since H_0' is connected, by Theorem 3.7, $H \cup H_0' \in \mathscr{C}$. By (C2), $(H \cup H_0')/H \in \mathscr{C}$. Thus $e \in E((H \cup H_0')/H)$ and $(H \cup H_0')/H \subseteq G/H$, so $e \notin E^*$. This contradiction shows $E^* \subseteq E'$, so $E^* = E'$. $\square$

Again, if the complete graph family $\mathscr{C}$ in Theorem 3.12 is not connected, the result can fail. Continuing with Example 3.2 and the graphs G, G_1, and G_2 defined in Example 3.11, the only candidate for H in Theorem 3.12 is K_1. But then $E^{**} = E(G)$, while $E^* = \emptyset$ since $G/H = G$. Thus $E^* \neq E^{**}$.

4. Free graph families

A graph family $\mathscr{S}$ is *free* if $\mathscr{S}$ satisfies the following three axioms:

(F1) $\mathscr{S}$ contains all edgeless graphs;
(F2) $\mathscr{S}$ is closed under taking subgraphs;
(F3) For any induced subgraph H of G, if $H \in \mathscr{S}$ and $G/H \in \mathscr{S}$, then $G \in \mathscr{S}$.

Notice that every free family is a subgraph lower ideal by (F2). Later in this section (Example 4.6), we will show that the family of all forests is free.

According to Lemma 3.2 of [5],

$$nK_1 \in \mathscr{S} \quad \text{for all } n \in \mathbf{Z}^+ \Leftrightarrow \mathscr{S}^O \subseteq \mathscr{S},$$

so every free and every complete graph family contains its kernel. Further, it is immediate from (F2) that any free graph family is also connected. The following lemma is a restatement of Lemmas 2.5 and 2.8.

Lemma 4.1. *For any graph family* $\mathscr{S}$, *each of* $\mathscr{S}^R$, $\mathscr{S}^C$, *and* $\mathscr{S}^H$ *satisfy* (F1) *and* $\mathscr{S}^R$ *also satisfies* (F2).

Lemma 4.2 ([5] 4.9). *Let $\mathscr{F}$ be a free graph family and suppose that $G \in \mathscr{F}$. Then $G \cup K_1 \in \mathscr{F}$.*

Proof. Since $G \in \mathscr{F}$, we note that $(G \cup K_1)/G = (\omega(G) + 1)K_1 \in \mathscr{F}$ by (F1). Thus, since G is an induced subgraph of $G \cup K_1$, it follows by (F3) that $G \cup K_1 \in \mathscr{F}$. $\square$

Thus all free families are stippled. Strengthening the $\mathscr{S}^C$ result of Theorem 3.8, we have

Lemma 4.3. *Let $\mathscr{F}$ be a free graph family. If $H \in \mathscr{F}^C$ and if the components of H are $H_1, H_2, \ldots, H_c$, then for any $I \subseteq \{1, 2, \ldots, c\}$, we have*

$$\bigcup_{i \in I} H_i \in \mathscr{F}^C.$$

Proof. For a contradiction, suppose that

$$\bigcup_{i \in I} H_i \notin \mathscr{F}^C.$$

Then there exists an edge subset $X \subset E(\bigcup_{i \in I} H_i)$ such that $H' = (\bigcup_{i \in I} H_i)/X \in \mathscr{F}$, and $E(H') \neq \emptyset$. By Lemma 4.2, $H' \cup (c - |I|)K_1 \in \mathscr{F}$. Therefore, the nontrivial graph

$$H/\left(X \cup \bigcup_{i \notin I} H_i\right) = H' \cup (c - |I|)K_1 \in \mathscr{F},$$

contrary to the assumption that $H \in \mathscr{F}^C$. $\square$

In [5, Theorem 4.6], it was claimed that, for any graph family $\mathscr{S}$ that is closed under contraction, $\mathscr{S}^R$ is a free graph family. This statement is false, as is shown by the following example:

Example 4.4. Let $\mathscr{S} = \mathscr{C}_{\omega > 1}$ and notice that $K_2 \in (\mathscr{C}_{\omega > 1})^R$. Let G be the path on three vertices and let H be either subgraph of G induced by an edge. Then both H and G/H are isomorphic to $K_2 \in (\mathscr{C}_{\omega > 1})^R$. But G is not in $(\mathscr{C}_{\omega > 1})^R$ by Theorem 3.3(a). Thus (F3) fails for $\mathscr{S}^R$ even though $\mathscr{S}$ is closed under contraction, and is even closed under CH-morphisms.

However, Theorem 4.6 of [5] can be proved for connected graph families that are closed under CH-morphisms:

Theorem 4.5 ([5] 4.6). *If $\mathscr{S}$ is a connected graph family which is closed under CH-morphisms, then $\mathscr{S}^R$ is a free graph family.*

Proof. By Lemma 4.1, (F1) and (F2) are satisfied. For a contradiction, suppose G is a graph and H an induced subgraph of G such that H and G/H are both in $\mathscr{S}^R$ but G

is not in $\mathcal{S}^R$. Then G contains a nontrivial subgraph $G'' \in \mathcal{S}$. Since $\mathcal{S}$ is connected, every component of G'' is in $\mathcal{S}$, so let G' be a nontrivial connected subgraph of G in $\mathcal{S}$.

Suppose $V(G') \subseteq V(H)$. Since H is induced, it follows that $G' \subseteq H$. Since $H \in \mathcal{S}^R$, by Lemma 2.5, $G' \in \mathcal{S}^R$. But then G' is a nontrivial graph in $\mathcal{S} \cap \mathcal{S}^R$, contrary to Lemma 2.8(a).

Hence $V(G') \nsubseteq V(H)$. Since G' is connected, it follows that $G'/(H \cap G')$ is nontrivial (where $G'/(H \cap G') = G'$ if $H \cap G'$ is edgeless). Since $\mathcal{S}$ is closed under CH-morphisms, and therefore under contractions, and since $G' \in \mathcal{S}$, we have $G'/(H \cap G') \in \mathcal{S}$. Moreover, $G'/(H \cap G')$ can be transformed by a CH-morphism to a loopless graph G^* with $E(G^*) = E(G'/(H \cap G'))$ such that $G^* \subseteq G/H$. Since $G'/(H \cap G') \in \mathcal{S}$ and $\mathcal{S}$ is closed under CH-morphisms, $G^* \in \mathcal{S}$. But then $G/H \notin \mathcal{S}^R$, contrary to assumption. This contradiction establishes (F3) for $\mathcal{S}^R$, so S^R is free. □

Example 4.6. The family of all forests is a free graph family.

Proof. Let $\mathcal{S} = \{K_1\} \cup \{\text{2-edge-connected graphs}\}$. $\mathcal{S}$ is connected by definition. Since CH-morphisms do not eliminate paths between vertices not identified under the CH-morphism, if G' is a nontrivial CH-morph of a 2-edge-connected graph G, then G' is 2-edge-connected graph. Thus $\mathcal{S}$ is closed under CH-morphisms. Thus $\mathcal{S}^R$ is free by Theorem 4.5. But $\mathcal{S}^R$ is the family of all forests by Example 2.3; hence the claim. □

Lemma 4.7. *Let $\mathcal{S}$ be a graph family. Any nontrivial graph in $(\mathcal{S}^R)^C$ has a nontrivial subgraph in $\mathcal{S}$.*

Proof. Let G be a nontrivial graph in $(\mathcal{S}^R)^C$. Then by definition, G has no nontrivial contraction in $\mathcal{S}^R$, and so $G \notin \mathcal{S}^R$. By the definition of $\mathcal{S}^R$, G has a nontrivial subgraph in $\mathcal{S}$. □

Lemma 4.7 suggests the question ‘When are $\mathcal{S}$ and $(\mathcal{S}^R)^C$ the same?’ While we do not have a complete answer to that question, the following lemma supplies a sufficient condition. The essence of the second paragraph of the proof appears in [5] and is included here for completeness.

Lemma 4.8 ([5] 4.11). *Let $\mathcal{C}$ be a complete graph family. Then $\mathcal{C} = (\mathcal{C}^R)^C$.*

Proof. Let $G \in (\mathcal{C}^R)^C$. Then G has no nontrivial contraction in $\mathcal{C}^R$, so any nontrivial contraction of G, including G itself, has a nontrivial subgraph in $\mathcal{C}$. Let H be a maximal nontrivial subgraph of G in $\mathcal{C}$. Suppose $H \neq G$. Then G/H has a nontrivial subgraph M in $\mathcal{C}$. Let H_M be the subgraph of G induced by $E(M)$. Then $(H_M \cup H)/H = M \cup nK_1$ for some integer n. Since $\mathcal{C}$ is stippled by Lemma 3.6 and $M \in \mathcal{C}$, we have $M \cup nK_1 \in \mathcal{C}$. Thus $H \in \mathcal{C}$, $H \subseteq H_M \cup H$, and $(H_M \cup H)/H \in \mathcal{C}$, so $H_M \cup H \in \mathcal{C}$ by (C3). But M is nontrivial in G/H, so $|E(H_M \cup H)| > |E(H)|$, contradicting the maximality of H. Thus $G = H$ and $G \in \mathscr{C}$, whence $(\mathscr{C}^R)^C \subseteq \mathscr{C}$.

Now suppose that $G \in \mathscr{C}$. By (C2), every contraction of G is in $\mathscr{C}$. Applying Lemma 2.8(a), G has no nontrivial contraction in $\mathscr{C}^R$. Thus, by definition, $G \in (\mathscr{C}^R)^C$. We have $\mathscr{C} \subseteq (\mathscr{C}^R)^C$, and equality follows. □

5. Examples

If $\mathscr{S}$ is a connected complete graph family, then by Corollary 3.5 and Theorem 4.5, $\mathscr{S}^R$ is free. We call $\mathscr{S}^R$ the *associated free graph family*.

Example 5.1. Let $k \in \mathbf{Z}^+$. The graph family

$$\mathscr{C} = \{nK_1 : n \in \mathbf{Z}^+\} \cup \{G : \text{each component } H \text{ of } G \text{ satisfies } \kappa'(H) > k\}$$

is a connected complete graph family, and the associated free graph family is $\mathscr{C}^R = \{G : \bar{\kappa}'(G) \leqslant k\}$, where

$$\bar{\kappa}'(G) = \max_{H \subseteq G} \kappa'(H).$$

Example 5.2. Define

$$\eta(G) = \min_{F \subseteq E(G)} \frac{|F|}{\omega(G-F)-1},$$

where the minimum is taken over all subsets $F \subseteq E(G)$ for which $\omega(G-F)$, the number of components of $G-F$, is at least 2. Tutte [17] and Nash-Williams [14] proved that $\lfloor \eta(G) \rfloor$ is the maximum number of edge-disjoint spanning trees of G, and Cunningham [7] showed that for $s, t \in \mathbf{Z}^+$, $\eta(G) \geqslant s/t$ if and only if G has a collection of s spanning trees such that at most t of them contain any given edge of G. Define

$$\gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}.$$

Nash-Williams [15] proved that $\lceil \gamma(G) \rceil$ is the edge-arboricity of G, and Catlin et al. [6] proved that for any $s, t \in \mathbf{Z}^+$, $\gamma(G) \leqslant s/t$ if and only if G has s spanning forests such that any given edge of G is contained in t of the s forests. For any rational $r \geqslant 1$, Catlin [5] proved that

$$\mathscr{C} = \{nK_1 : n \in \mathbf{Z}^+\} \cup \{G : \eta(G) \geqslant r\}$$

is a complete graph family, that $\mathscr{C}^R = \{nK_1 : n \in \mathbf{Z}^+\} \cup \{G : \gamma(G) < r\}$ is the associated free graph family, and that

$$\gamma(G) = \max_{H \subseteq G} \eta(H).$$

Note that $\mathscr{C}$ is also connected.

Example 5.3. It follows from the definition of $\mathscr{S}^R$ that whenever a graph family $\mathscr{S}$ is defined by

$$\mathscr{S} = \{nK_1 : n \in \mathbf{Z}^+\} \cup \{G : \text{each component } H \text{ of } G \text{ satisfies } \mu(H) \geqslant r\}$$

for some graph invariant μ (e.g., $\mu = \kappa'$ or $\mu = \eta$), then the associated graph family $\mathscr{S}^R$ is $\{nK_1 : n \in \mathbf{Z}^+\} \cup \{G : \bar{\mu}(G) < r\}$, where $\bar{\mu}(G) = \max_{H \subseteq G} \mu(H)$.

However, some important graph families are defined in the absence of such a function μ, e.g., the family $\mathscr{SL}$ of supereulerian graphs. By Theorem 3.1, the kernel $\mathscr{SL}^O$ of $\mathscr{SL}$ is a complete graph family.

A graph G is *collapsible* if for every even subset $V_i \subseteq V(G)$, G has a subgraph H_i with $V(H_i = V_i$ such that every vertex in H_i has odd degree in H_i and $G - E(H_i)$ is connected. Define $\mathscr{CL}$ to be the family of graphs whose components are collapsible. In [5] Catlin proved that $\mathscr{CL}$ is a complete graph family and that $\mathscr{CL} \subseteq \mathscr{SL}^O$. Also, he showed that any graph G in $\mathscr{CL}^R$ (and hence any graph in $(\mathscr{SL}^O)^R$) is nK_1 or has $\gamma(G) < 2$ and girth at least 4. The actual value of $\mathscr{SL}^O$ is unknown, but we conjecture that it is $\mathscr{CL}$. Both are connected complete graph families.

For any graph family $\mathscr{S}$ and any $k \in \mathbf{N}$, we say that a graph G is *at most k edges short of being in $\mathscr{S}$* if $\mathscr{S}$ contains a graph G' having a set E' of at most k edges, such that $G = G' - E'$.

Example 5.4 (*Lai and Lai* [13]). Let $k \in \mathbf{N}$. If $\mathscr{C}$ is a complete graph family and if $\mathscr{C}(k)$ is the family of graphs that are at most k edges short of being in $\mathscr{C}$, then $\mathscr{C} = (\mathscr{C}(k))^O$.

In [5], Catlin proved the two special cases of this result where $\mathscr{C}$ is the complete graph family of Example 5.1 or Example 5.2.

Example 5.5. Let A be a finite abelian group with $|A| \geqslant 3$. It is routine to verify that the family $[A]$ of all A-connected graphs is a connected complete graph family, and that if $|A| = k \geqslant 3$, then $[A] \subseteq (F_k)^O$. Catlin has shown [3] that $\mathscr{CL} \subset (F_4)^O$. Recently [11] this was improved to $\mathscr{CL} \subseteq [A] \subset (F_4)^O$, for any group A with $|A| = 4$. The last containment is proper: Catlin indicated in [3] that the 4-cycle is in $(F_4)^O$, but it is easy to see that the 4-cycle is not in $[A]$ for any abelian group A with $|A| = 4$.

6. Connections between graph operations

The next theorem is Theorem 4.10 of [5]. Unfortunately, the proof provided in that paper is faulty. Specifically, in that proof, when isolated vertices are added to graph H_0 to form graph J, there is an assumption made that if the edges of a subgraph survive a contraction, so does the subgraph. But this is not so (distinct vertices of the subgraph may have been coalesced into a single vertex), and so the graph H_0 of that proof cannot necessarily be converted into graph J as claimed by adding isolated vertices. Nevertheless, the theorem is correct, as the following lemmas and proof of Theorem 6.3 show.

Lemma 6.1. *Let $\mathscr{F}$ be a free graph family. If $\mathscr{F}$ contains a nontrivial graph, then $K_2 \in \mathscr{F}$. Moreover, if $\mathscr{F}$ consists of all edgeless finite graphs, then $\mathscr{F}^C$ is complete, $\mathscr{F}^C$ consists of all finite graphs, and $\mathscr{F} = (\mathscr{F}^C)^R$.*

Proof. The first part of the lemma is a consequence of (F2). Suppose $\mathscr{F}$ consists of all edgeless finite graphs. Since no graph can have a nontrivial contraction in $\mathscr{F}$, every finite graph is in $\mathscr{F}^C$ by definition. That $\mathscr{F}^C$ satisfies (C1) and (C2) is now immediate. Also, let G be any finite graph. Then, given $H \subseteq G$ with $H \in \mathscr{F}^C$ and $G/H \in \mathscr{F}^C$, $G \in \mathscr{F}^C$ since G is finite. Thus (C3) follows, and $\mathscr{F}^C$ is complete.

Since $\mathscr{F}^C$ is the set of all finite graphs, by definition $(\mathscr{F}^C)^R$ can have no nontrivial members. Thus $\mathscr{F} = (\mathscr{F}^C)^R$ by Lemma 2.8(a). $\square$

The next lemma supplies a key step in the proof of Theorem 6.3 In this lemma, we stretch the meaning of $H + e$ by allowing an edge e not in H to be added even if its ends are not in H. In such a case the end or ends not in H are also added.

Lemma 6.2 ([5] 4.7). *Let $\mathscr{F}$ be a free graph family. Let H be a connected graph. Suppose that $H \in \mathscr{F}^C$ and suppose that e is a nonloop edge not in H. If $(H + e)/H \in \mathscr{F}^C$, then $H + e \in \mathscr{F}^C$.*

Proof. This is immediate from Lemma 6.1 if $\mathscr{F}$ has no nontrivial graphs, so we may suppose $\mathscr{F}$ includes a nontrivial graph. Also note that $\mathscr{F}^C$ satisfies (C2) by Lemma 2.6. Let M denote the subgraph of $H + e$ induced by edge e. Then M is isomorphic to K_2 and is in $\mathscr{F}$ by (F2). For a contradiction, suppose H and e are chosen so that $|E(H + e)|$ is as small as possible with $H + e \notin \mathscr{F}^C$. By the definition of $\mathscr{F}^C$, we may select a maximal set $X \subset E(H + e)$ such that $(H + e)/X \in \mathscr{F}$ and is nontrivial.

Case 1: Suppose $|V(M) \cap V(H)| \leqslant 1$. Then $(H + e)/H = M \cup tK_1$ for some $t \in \{0, 1\}$. If $e \in X$, then $H/(X - \{e\}) = (H + e)/X \in \mathscr{F}$, contrary to $H \in \mathscr{F}^C$ since $(H + e)/X$ is nontrivial.

Suppose $e \notin X$. If $E((H + e)/X) - \{e\} \neq \emptyset$, then $H/X = (H + e)/X - e$ is nontrivial, and it is in $\mathscr{F}$ by (F2), since $(H + e)/X \in \mathscr{F}$. This is a contradiction of $H \in \mathscr{F}^C$.

Thus we may suppose $E((H + e)/X) = \{e\}$. Since X is maximal, it follows that $X = E(H)$ and $M \cup tK_1 = (H + e)/X = (H + e)/H \in \mathscr{F}^C$ by hypothesis, where $t \in \{0, 1\}$. But $M \in \mathscr{F}$ as observed above, so $M \cup tK_1 \in \mathscr{F}$ by Lemma 4.2. Thus $\mathscr{F} \cap \mathscr{F}^C$ includes the nontrivial graph M, contrary to Lemma 2.8(b).

Case 2: Thus we may suppose that $e = uv$ and both $u, v \in V(H)$. For a contradiction, suppose that $e \notin X$. Since X is maximal, we have that $e \in E((H + e)/X$. Since $(H + e)/X \in \mathscr{F}$, by (F2) we have $(H + e)/X - e \in \mathscr{F}$. Thus $H/X = ((H + e) - e)/X = (H +$

$e)/X-e\in\mathscr{F}$. But we know $H\in\mathscr{F}^C$, so by (C2) (Lemma 2.6), $H/X\in\mathscr{F}^C$. By Lemma 2.8(b), then, $E(H/X)=\emptyset$, so $E((H+e)/X)=\{e\}$. Moreover, $X=E(H)$ since X is maximal. But H is connected, so $E((H+e)/X)=E((H+e)/E(H))=\emptyset$, a contradiction.

Thus we may suppose $e\in X$. Suppose there is an edge $e'\in X-\{e\}$, so $e'\in E(H)$. Let $G'=(H+e)/e'$ and let $H'=H/e'$. Either the ends of e are left different by the contraction of e', so that $G'=H'+e$, or e and e' are parallel in $H+e$, in which case e is a loop in G' and so is erased, leaving $G'=H'$. Since $H\in\mathscr{F}^C$, by (C2) we have $H'\in\mathscr{F}^C$. Also, $G'/H'=((H+e)/e')/(H/e')=K_1\in\mathscr{F}^C$. By the minimality of $|E(H+e)|$, it follows that $G'\in\mathscr{F}^C$. Thus $G'/(X-\{e'\})\in\mathscr{F}^C$ by (C2). But $G'/(X-\{e'\})=((H+e)/e')/(X-\{e'\})=(H+e)/X$, so $(H+e)/X\in\mathscr{F}^C$. Since X was chosen so that $(H+e)/X\in\mathscr{F}$ and nontrivial, this is impossible. Hence $X=\{e\}$.

Suppose H has an edge e'' parallel to e. Then $H/e''\in\mathscr{F}^C$ by (C2) since $H\in\mathscr{F}^C$. But in this case, $(H+e)/e=H/e''$, so $(H+e)/e=(H+e)/X\in\mathscr{F}^C\cap\mathscr{F}$. Since $(H+e)/X$ is nontrivial, this is a contradiction of Lemma 2.8(b).

Thus we may suppose $X=\{e\}$ and M is an induced subgraph of $H+e$.

Since F includes a nontrivial graph, by (F2) we have $K_2\in\mathscr{F}$. Since $M\cong K_2$, this shows that $M\in\mathscr{F}$. But we assumed that $(H+e)/M=(H+e)/X\in\mathscr{F}$. By (F3) it follows that $H+e\in\mathscr{F}$. From this, (F2) gives us $H\in\mathscr{F}$. But H is nontrivial and is now in $\mathscr{F}^C\cap\mathscr{F}$, contrary to Lemma 2.8(b). This final contradiction proves Lemma 6.2. $\square$

Theorem 6.3 ([5] 4.10). *If $\mathscr{F}$ is a free graph family, then $\mathscr{F}^C$ is a complete graph family and $\mathscr{F}=(\mathscr{F}^C)^R$.*

Proof. This theorem is given by Lemma 6.1 if $\mathscr{F}$ has no nontrivial graphs. If $\mathscr{F}$ has a nontrivial graph, then $\mathscr{F}^C$ satisfies (C1) by the definition of $\mathscr{F}^C$ and (C2) by Lemma 2.6. Thus we want to show

(A) if $H\subseteq G$, $H\in\mathscr{F}^C$ and $G/H\in\mathscr{F}^C$, then $G\in\mathscr{F}^C$.
For a contradiction, we assume that
(B) H and G form a counterexample to (A) with $|E(G)|$ smallest;
thus $H\subseteq G$, $H\in\mathscr{F}^C$ and $G/H\in\mathscr{F}^C$, and yet $G\notin\mathscr{F}^C$.

We note that $|E(G)|>0$, for otherwise $G\in\mathscr{F}^C$ by (C1). Since $G\notin\mathscr{F}^C$, there is a maximal edge set $X\subset E(G)$ such that $G_0=G/X\in\mathscr{F}$ and G_0 is nontrivial. Since X is maximal, $E(G_0)=E(G)-X$.

If $E(H)=\emptyset$, then $G/H=G$, while $G/H\in\mathscr{F}^C$, so that $G\in\mathscr{F}^C$, contrary to assumption. Thus $E(H)\neq\emptyset$.

If $X=\emptyset$, then $G=G/X$, so $G\in\mathscr{F}$. By (F2) it follows that $H\in\mathscr{F}$. But this contradicts Lemma 2.8(b) since $H\in\mathscr{F}^C$ and $E(H)\neq\emptyset$. Thus $X\neq\emptyset$.

Suppose $E(H)\cap X\neq\emptyset$, and let $e\in E(H)\cap X$. Let $G_1=G/e$. Since $H\in\mathscr{F}^C$, by (C2) we have $H/e\in\mathscr{F}^C$. Moreover, $G_1/(H/e)=(G/e)/(H/e)=G/H$. By (B), $G/H\in\mathscr{F}^C$, so $G_1/(H/e)\in\mathscr{F}^C$. Since $H\subseteq G$ by (B), we also have $H/e\subseteq G/e=G_1$. By these observations and the minimality of G as a counterexample, we conclude that $G_1\in\mathscr{F}^C$. By

(C2) it follows that $G_1/(X-\{e\}) \in \mathscr{F}^C$. But $G_1/(X-\{e\})=(G/e)/(X-\{e\})=G/X=G_0 \in \mathscr{F}$, and G_0 is nontrivial as observed before. Thus the nontrivial graph $G_1/(X-\{e\})=G/X \in \mathscr{F}^C \cap \mathscr{F}$, contrary to Lemma 2.8(b). Hence $E(H) \cap X=\emptyset$.

Suppose H is connected. By our previous observations, we may choose $e \in X$, and we know $e \notin E(H)$. Let $G_2=G/e$ and let $M=G[e]$.

Suppose $|V(M) \cap V(H)| \leqslant 1$. Then $H \subseteq G_2$. By (B), $H \in \mathscr{F}^C$. Moreover, $G_2/H=(G/e)/H=(G/H)/e$. Since $G/H \in \mathscr{F}^C$ by (B), by (C2) $(G/H)/e \in \mathscr{F}^C$. Thus $G_2/H \in \mathscr{F}^C$. Since G is a smallest counterexample to the theorem, we conclude $G_2 \in \mathscr{F}^C$. But then $G_2/(X-\{e\}) \in \mathscr{F}^C$ by (C2). Thus $G_2/(X-\{e\})=(G/e)/(X-\{e\})=G/X \in \mathscr{F}^C$. Since $G/X \in \mathscr{F}$ as well and is nontrivial, this is impossible by Lemma 2.8(b).

Thus $|V(M) \cap V(H)|=2$. If there is an edge $e' \in E(H)$ parallel to e, then we may replace X with $X'=(X-\{e\}) \cup \{e'\}$. Now $X' \cap E(H) \neq \emptyset$, and we get the same contradiction we got before. Thus no such edge e' exists.

We now have that $(H+e)/H=K_1 \in \mathscr{F}^C$. By Lemma 6.2, we get $H+e \in \mathscr{F}^C$, and (C2) gives us $(H+e)/e \in \mathscr{F}^C$. But $(H+e)/e \subseteq G_2$. By (C2) and $G/H \in \mathscr{F}^C$, we have $G_2/((H+e)/e)=(G/e)/((H+e)/e)=(G/e)/H=(G/H)/e$. But $G/H \in \mathscr{F}^C$ by assumption, so (C2) gives $(G/H)/e \in \mathscr{F}^C$. Since G is a smallest counterexample and $|E(G_2)|<|E(G)|$, it follows that $G_2 \in \mathscr{F}^C$. But $G_0=G/X=(G/e)/(X-\{e\})=G_2/(X-\{e\}) \in \mathscr{F}^C$ by (C2). Thus $G_0 \in \mathscr{F}^C \cap \mathscr{F}$, contrary to Lemma 2.8(b). Thus the first part of our theorem is true if H has only one component.

Now suppose H is the union of disjoint components $H_1, H_2, \ldots, H_c$ for some integer $c \geqslant 1$. Let $G^i=G/(\bigcup_{1 \leqslant \ell \leqslant i} H_\ell)$ for each $i \in \{1,2,\ldots,c\}$.

Next, we prove $G^i \in \mathscr{F}^C$ for each $i \geqslant 1$. Note that $G^c=G/(\bigcup_{1 \leqslant \ell \leqslant c} H_\ell)=G/H \in \mathscr{F}^C$ by assumption. Suppose that we have $G^{c-i} \in \mathscr{F}^C$ for some $i \geqslant 0$. Since the components of H are vertex disjoint, $H_{c-i} \subseteq G^{c-i-1}$. Moreover, $G^{c-i-1}/H_{c-i}=G^{c-i} \in \mathscr{F}^C$, $H_{c-i} \in \mathscr{F}^C$ by Lemma 4.3, and $H_{c-i} \subseteq G^{c-i-1}$. But $|E(G^{c-i-1})|<|E(G)|$, so $G^{c-i-1} \in \mathscr{F}^C$ since G is a smallest counterexample.

We conclude that $G^1 \in \mathscr{F}^C$. But then we have $G/H_1 \in \mathscr{F}^C$, $H_1 \subseteq G$, and $H_1 \in \mathscr{F}^C$. Since H_1 is connected, the earlier part of this proof shows that $G \in \mathscr{F}^C$. Thus (B) fails and $\mathscr{F}^C$ is complete.

Finally we prove that $\mathscr{F}=(\mathscr{F}^C)^R$. By the observation made at the beginning of this proof, we may suppose $\mathscr{F}$ includes a nontrivial graph.

Suppose $G \in \mathscr{F}$ and suppose $G \notin (\mathscr{F}^C)^R$. Then by the definition of the operator R, G has a nontrivial subgraph $H \in \mathscr{F}^C$. Since $G \in \mathscr{F}$, by (F2) we have $H \in \mathscr{F}$, so $H \in \mathscr{F}^C \cap \mathscr{F}$. By Lemma 2.8(b), H has no edges, a contradiction. Thus

$$\mathscr{F} \subseteq (\mathscr{F}^C)^R. \tag{3}$$

Next, suppose G is a graph which minimizes $|V(G)|+|E(G)|$ over the graphs in $(\mathscr{F}^C)^R-\mathscr{F}$. Since $\mathscr{F}$ includes all edgeless graphs, G is nontrivial. By the definition of operation R, $G \notin \mathscr{F}^C$.

Case A: Suppose G is disconnected. Let H be a component of G and let $H'=G-H$. By Lemma 2.5, we have $H, H' \in (\mathscr{F}^C)^R$. Hence by the minimality of G, both H and H' are in $\mathscr{F}$. Let e be an edge joining a vertex of H to a vertex of H', let $G'=H+e$, and let $G'' = (H \cup H') + e = G + e$. Then $G'[e]$ and H are vertex-induced subgraphs of G'. Further, $G'[e] \in \mathscr{F}$ because $\mathscr{F}$ includes a nontrivial graph as noted before, and $H = G'/e \in \mathscr{F}$ by assumption. Hence $G' \in \mathscr{F}$ by (F3). But $H' \in \mathscr{F}$, H' is an induced subgraph of G'', and $G''/H' = G' \cup nK_1$ for some integer n, and $G' \cup nK_1 \in \mathscr{F}$ by Lemma 4.2. Thus $G'' \in \mathscr{F}$ by (F3). By (F2), we conclude that $G = G'' - e \in \mathscr{F}$, a contradiction.

Case B: Thus we may suppose G is connected. Since $G \notin \mathscr{F}^C$, there is a nontrivial contraction G_0 of G in $\mathscr{F}$. Since $G \notin \mathscr{F}$, $G \neq G_0$; moreover, the connectedness of G ensures the connectedness of G_0. Since G_0 is nontrivial, $c := |V(G_0)| \geqslant 2$. Let the vertices of G_0 be $v_1, \ldots, v_c$ and, for each $i \in \{1, 2, \ldots, c\}$, let H_i be the maximal subgraph of G which contracts to v_i in forming G_0 from G. Form graphs $G_1, \ldots, G_c$ in order by forming G_i from G_{i-1} by replacing v_i with H_i and connecting the v_i-ends of the edges of G_0 which meet v_i to their ends in G. For each i, H_i is an induced subgraph of G_i. Since $H_i \in (\mathscr{F}^C)^R$ by Lemma 2.5, the minimality of G in $(\mathscr{F}^C)^R - \mathscr{F}$ gives us $H_i \in \mathscr{F}$. To begin an inductive argument, we note that $G_0 \in \mathscr{F}$. At each stage, since $G_{i-1} \in \mathscr{F}$, we have $G_i \in \mathscr{F}$ by (F3). Thus $G_c \in \mathscr{F}$. But $G_c = G$, contradicting the choice of G, so $(\mathscr{F}^C)^R \subseteq \mathscr{F}$. Combining this with (3) completes the proof of Theorem 6.3. $\square$

The next lemma extends Lemma 2.9.

Lemma 6.4. *For any free graph family* $\mathscr{F}$, $\mathscr{F}^H = \mathscr{F}^C$.

Proof. Suppose that $\mathscr{F}$ is free, let $G \in \mathscr{F}^C$, and suppose by way of contradiction that $G \notin \mathscr{F}^H$. Then G has a CH-morphism to a nontrivial graph $G_0 \in \mathscr{F}$. By Theorem 6.3, $\mathscr{F}^C$ is complete and thus by Corollary 3.5, $\mathscr{F}^C$ is closed under CH-morphisms, and so $G_0 \in \mathscr{F}^C$. But then the nontrivial graph G_0 is in $\mathscr{F} \cap \mathscr{F}^C$, contrary to Lemma 2.8(b). Hence, $\mathscr{F}^C \subseteq \mathscr{F}^H$ when $\mathscr{F}$ is free. Applying Lemma 2.9, $\mathscr{F}^C = \mathscr{F}^H$. $\square$

Definition. A graph family $\mathscr{S}$ is *antistippled* if, for any $G \in \mathscr{S}$ with an isolated vertex x, the graph $G - x$ is also in $\mathscr{S}$.

Lemma 6.5. *Let* $\mathscr{S}$ *be a graph family.*

(a) *If* $\mathscr{S}$ *is connected and complete, then* $\mathscr{S}$ *is antistippled.*
(b) *If* $\mathscr{S}$ *is antistippled, then* $\mathscr{S}^R$ *is stippled.*

Proof. Let $G \in \mathscr{S}$ and let x be an isolated vertex of G. Let $H = G - x$, and suppose the components of H are $C_1, C_2, \ldots, C_r$. Since $\mathscr{S}$ is connected, and since each component of H is a component of G, $C_1, C_2, \ldots, C_r \in \mathscr{S}$. Applying Theorem 3.7 $r-1$ times, $H = \bigcup_{1 \leqslant i \leqslant r} C_i \in \mathscr{S}$. Thus part (a) of the lemma is proved.

Now let $G \in \mathscr{S}^R$. Then G has no nontrivial subgraph in $\mathscr{S}$. Suppose $G \cup K_1 \notin \mathscr{S}^R$. Then $G \cup K_1$ has a nontrivial subgraph $H \in \mathscr{S}$. If $\{x\} = V(G \cup K_1) - V(G)$, and if $x \notin V(H)$, then H is a nontrivial subgraph of G in $\mathscr{S}$, a contradiction. So $x \in V(H)$.

But x is isolated in $H \in \mathscr{S}$, so $H - x \in \mathscr{S}$ by the antistippled property. This is a contradiction, and part (b) of the lemma follows. $\square$

Note that by Lemma 4.2 and (F2), every free family is both stippled and antistippled.

Theorem 6.6. *For any graph family* $\mathscr{S}$,

(a) *if* $\mathscr{S}$ *is antistippled, then* $(\mathscr{S}^R)^H$ *is connected and complete, and* $(\mathscr{S}^R)^H$ *is contained in every complete graph family containing* $\mathscr{S}$; *and*
(b) *if* $\mathscr{S}$ *is stippled, then* $(\mathscr{S}^H)^R$ *is free, and* $(\mathscr{S}^H)^R$ *is contained in every free graph family containing* $\mathscr{S}$.

Proof. Let $\mathscr{S}$ be a graph family.

(a) By Lemma 2.5, $\mathscr{S}^R$ is closed under taking subgraphs. By Lemma 6.5(b), $\mathscr{S}^R$ is stippled, so by Theorem 3.9, $(\mathscr{S}^R)^H$ is connected and complete. There is a complete graph family containing $\mathscr{S}$: the family of all graphs. Now let $\mathscr{C}$ be a complete graph family with $\mathscr{S} \subseteq \mathscr{C}$. Apply (a) and (c) of Lemma 2.4 successively to get

$$\mathscr{C}^R \subseteq \mathscr{S}^R; \quad \text{and} \quad (\mathscr{S}^R)^H \subseteq (\mathscr{C}^R)^H.$$

Since $\mathscr{C}$ is complete, $\mathscr{C} = (\mathscr{C}^R)^C$ by Lemma 4.8. Hence, $(\mathscr{C}^R)^H \subseteq \mathscr{C}$ by Lemma 2.9, and so $(\mathscr{S}^R)^H \subseteq \mathscr{C}$, and (a) is proved.

(b) By Lemma 2.7, $\mathscr{S}^H$ is closed under CH-morphisms, and by Theorem 3.8, $\mathscr{S}^H$ is connected. Thus, by Theorem 4.5, $(\mathscr{S}^H)^R$ is free. Now let $\mathscr{F}$ be a free graph family with

$$\mathscr{S} \subseteq \mathscr{F}.$$

Since the family of all graphs is a free graph family, such a graph family $\mathscr{F}$ exists. By Lemma 2.4(c), $\mathscr{F}^H \subseteq \mathscr{S}^H$, and so by Lemma 2.4(a), $(\mathscr{S}^H)^R \subseteq (\mathscr{F}^H)^R$. But $\mathscr{F}^H = \mathscr{F}^C$ by Lemma 6.4. Thus $(\mathscr{F}^H)^R = (\mathscr{F}^C)^R = \mathscr{F}$ by Theorem 6.3. Hence we have $(\mathscr{S}^H)^R \subseteq \mathscr{F}$. $\square$

A consequence of Theorem 6.6(a) is that if $\mathscr{S}$ is antistippled and $\mathscr{S} \subseteq \mathscr{C}_{\omega>1}$, then $(\mathscr{S}^R)^H = \{nK_1 : n \in \mathscr{Z}\}$.

Corollary 6.7. *For any graph family* $\mathscr{S}$, *the following are equivalent*:

(a) $\mathscr{S}$ *is antistippled and* $\mathscr{S} = (\mathscr{S}^R)^H$;
(b) $\mathscr{S}$ *is a connected complete graph family.*

Proof. By Theorem 6.6(a), (a) $\Rightarrow$ (b). Suppose (b). Then by Lemma 6.5(a), $\mathscr{S}$ is antistippled. By Lemma 4.8, $\mathscr{S} = (\mathscr{S}^R)^C$. But $\mathscr{S}^R$ is free by Corollary 3.5 and Theorem 4.5, so $\mathscr{S} = (\mathscr{S}^R)^C = (\mathscr{S}^R)^H$ by Lemma 6.4. Thus (b) $\Rightarrow$ (a). $\square$

Corollary 6.8. *For any graph family* $\mathscr{S}$, *the following are equivalent*:

(a) $\mathscr{S}$ *is stippled and* $\mathscr{S} = (\mathscr{S}^H)^R$;
(b) $\mathscr{S}$ *is a free graph family.*

Proof. By Theorem 6.6(b), (a) $\Rightarrow$ (b). By Lemma 4.2, Theorem 6.3 and Lemma 6.4, (b) $\Rightarrow$(a). $\square$

The next theorem could have been included in Section 2, but its corollaries depend on Corollaries 6.7 and 6.8, so it is here instead.

Theorem 6.9. *Let $\mathscr{S}$ be a family of graphs*:

(a) *If $\mathscr{S}$ is closed under CH-morphisms, then* $\mathscr{S} \subseteq (\mathscr{S}^R)^H$;
(b) *If $\mathscr{S}$ is closed under taking subgraphs, then* $\mathscr{S} \subseteq (\mathscr{S}^H)^R$.

Proof. Let $\mathscr{S}$ be a graph family:

(a) Suppose $\mathscr{S}$ is closed under CH-morphisms, and let $G \in \mathscr{S}$. By way of contradiction, suppose $G \notin (\mathscr{S}^R)^H$. Then some nontrivial CH-morph G_0 of G is in $\mathscr{S}^R$. Since $G \in \mathscr{S}$ and $\mathscr{S}$ is closed under CH-morphisms, $G_0 \in \mathscr{S}$. Hence, G_0 is a nontrivial graph in $\mathscr{S}^R \cap \mathscr{S}$, a contradiction of Lemma 2.8(a).

(b) Suppose $\mathscr{S}$ is closed under taking subgraphs, and let $G \in \mathscr{S}$. By way of contradiction, suppose $G \notin (\mathscr{S}^H)^R$. Hence, G has a nontrivial subgraph $G_0 \in \mathscr{S}^H$. Since $G \in \mathscr{S}$ and $\mathscr{S}$ is closed under taking subgraphs, $G_0 \in \mathscr{S}$. Hence, G_0 is a nontrivial graph in $\mathscr{S}^H \cap \mathscr{S}$, contrary to Lemma 2.8(c). $\square$

Theorem 6.9 cannot be readily strengthened since $\mathscr{S} \subseteq (\mathscr{S}^R)^H$ and $\mathscr{S} \subseteq (\mathscr{S}^H)^R$ of Theorem 6.9 are not true if $\mathscr{S} = \{K_3\}$. To see this, we have

$$\mathscr{S}^R = \{K_3\text{-free graphs}\}.$$

Therefore, a nontrivial graph $G \in (\mathscr{S}^R)^H$ if and only if no nontrivial CH-morph of G is K_3-free. But the 2-cycle is a CH-morph of $G = K_3$. Hence, $K_3 \notin (\mathscr{S}^R)^H$, and $\mathscr{S} \nsubseteq (\mathscr{S}^R)^H$. Next, note that

$$\mathscr{S}^H = \{G : K_3 \text{ is not a CH-morph of } G\}$$

Hence, $K_2 \in \mathscr{S}^H$. Now

$$(\mathscr{S}^H)^R = \{G' : \text{no nontrivial subgraph of } G' \text{is in } \mathscr{S}^H\}.$$

Since $K_2 \in \mathscr{S}^H$, this implies

$$(\mathscr{S}^H)^R = \{\text{edgeless graphs}\}.$$

Therefore, $K_3 \notin (\mathscr{S}^H)^R$, and $\mathscr{S} \nsubseteq (\mathscr{S}^H)^R$.

Corollary 6.10. *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be graph families. If $\mathscr{S}_1$ is closed under taking subgraphs and if $\mathscr{S}_2$ is free, then*

$$\mathscr{S}_1 \subseteq \mathscr{S}_2 \Leftrightarrow \mathscr{S}_2^H \subseteq \mathscr{S}_1^H.$$

Proof. '$\Rightarrow$' is Lemma 2.4(c). To prove '$\Leftarrow$', suppose $\mathscr{S}_2^H \subseteq \mathscr{S}_1^H$. Then by Theorem 6.9(b), Lemma 2.4(a), and Corollary 6.8,

$$\mathscr{S}_1 \subseteq (\mathscr{S}_1^H)^R \subseteq (\mathscr{S}_2^H)^R = \mathscr{S}_2. \qquad \square$$

Corollary 6.11 ([5] 4.14). *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be graph families. If $\mathscr{S}_1$ is closed under CH-morphisms and if $\mathscr{S}_2$ is connected and complete, then*

$$\mathscr{S}_1 \subseteq \mathscr{S}_2 \Leftrightarrow \mathscr{S}_2^R \subseteq \mathscr{S}_1^R.$$

Proof. '$\Rightarrow$' is Lemma 2.4(a). To prove '$\Leftarrow$', suppose $\mathscr{S}_2^R \subseteq \mathscr{S}_1^R$. Then by Theorem 6.9(a), Lemma 2.4(c), and Corollary 6.7,

$$\mathscr{S}_1 \subseteq (\mathscr{S}_1^R)^H \subseteq (\mathscr{S}_2^R)^H = \mathscr{S}_2. \qquad \Box$$

In [5], Paul Catlin proved a similar result for operation C:

Theorem 6.12 (Catlin [5, 4.13]). *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be graph families. If $\mathscr{S}_1$ and $\mathscr{S}_2$ are each free, then*

$$\mathscr{S}_1 \subseteq \mathscr{S}_2 \Leftrightarrow (\mathscr{S}_2)^C \subseteq (\mathscr{S}_1)^C.$$

Theorems 6.6 and 6.9 lead to an interesting view of free and connected complete graph families as members of lattices, as follows:

For any two connected complete families $\mathscr{C}_1$ and $\mathscr{C}_2$, define the *meet*

$$\mathscr{C}_1 \wedge \mathscr{C}_2 = \mathscr{C}_1 \cap \mathscr{C}_2.$$

It follows routinely from the definition that $\mathscr{C}_1 \wedge \mathscr{C}_2$ is a connected complete graph family. Define the *join* $\mathscr{C}_1 \vee \mathscr{C}_2$ to be the intersection of all connected complete graph families containing $\mathscr{C}_1 \cup \mathscr{C}_2$. By Corollary 3.5 and Theorem 6.9(a),

$$\mathscr{C}_1 \cup \mathscr{C}_2 \subseteq ((\mathscr{C}_1 \cup \mathscr{C}_2)^R)^H.$$

But all connected complete families are antistippled by Lemma 6.5(a), and consequently $\mathscr{C}_1 \cup \mathscr{C}_2$ is easily shown to be antistippled. Thus, by Theorem 6.6(a), $((\mathscr{C}_1 \cup \mathscr{C}_2)^R)^H$ is connected and complete. Moreover, it is in the intersection of all complete graph families containing $\mathscr{C}_1 \cup \mathscr{C}_2$. Since it is one of these complete graph families,

$$\mathscr{C}_1 \vee \mathscr{C}_2 = ((\mathscr{C}_1 \cup \mathscr{C}_2)^R)^H$$

and $\mathscr{C}_1 \vee \mathscr{C}_2$ is connected and complete. Thus the connected complete graph families form a lattice.

Dually, for free graph families $\mathscr{F}_1$ and $\mathscr{F}_2$, define the *meet*

$$\mathscr{F}_1 \wedge \mathscr{F}_2 = \mathscr{F}_1 \cap \mathscr{F}_2.$$

It is routine to show that $\mathscr{F}_1 \wedge \mathscr{F}_2$ is free. Define the *join* $\mathscr{F}_1 \vee \mathscr{F}_2$ to be the intersection of all free graph families containing $\mathscr{F}_1 \cup \mathscr{F}_2$. By Lemma 4.2, $\mathscr{F}_1 \cup \mathscr{F}_2$ is stippled. Thus, by Theorems 6.9(b) and 6.6(b),

$$\mathscr{F}_1 \vee \mathscr{F}_2 = ((\mathscr{F}_1 \cup \mathscr{F}_2)^H)^R.$$

Thus the free graph families also form a lattice. $\Box$

7. A special case

Define

$$\bar{\delta}(G) = \max_{H \subseteq G} \delta(H); \quad \text{and} \quad \bar{\kappa}'(G) = \max_{H \subseteq G} \kappa'(H),$$

so, for example, if $K_3 \cup K_4$ is the disjoint union of K_3 and K_4, then $\bar{\kappa}'(K_3 \cup K_4) = 3$, and $\bar{\delta}(K_3 \cup K_4) = 3$. A graph G such that $\bar{\delta}(G) \leqslant k$ is called *k-degenerate*.

Theorem 7.1. *Let $k \in \mathscr{Z}$. If $\mathscr{S}_k$ denotes $\{G' : \bar{\delta}(G') \leqslant k\}$*, then

$$(\mathscr{S}_k^H)^R = \{G : \bar{\kappa}'(G) \leqslant k\}.$$

Proof. Let $\mathscr{T}_k := \{G : \bar{\kappa}'(G) \leqslant k\}$. Suppose $G \in (\mathscr{S}_k^H)^R$. Then for any subgraph H of G, $H \notin \mathscr{S}_k^H$. If $\kappa'(H) > k$, then every nontrivial CH-morph H' of H satisfies $\kappa'(H') > k$, and so H has no CH-morph H' with $\bar{\delta}(H') \leqslant k$. Hence $H \in \mathscr{S}_k^H$, a contradiction. Thus $\kappa'(H) \leqslant k$ for all $H \subseteq G$, $G \in \mathscr{T}_k$, and so $(\mathscr{S}_k^H)^R \subseteq \mathscr{T}_k$.

For the other containment, let $G \in \mathscr{T}_k$, and let H be a nontrivial subgraph of G. For a contradiction, suppose $H \in \mathscr{S}_k^H$. Then H has no CH-morph in $\mathscr{S}_k$. We describe such a CH-morph, arriving at a contradiction: Since H is nontrivial, it has a nontrivial component C_0. Let the other components of H be $C_1, C_2, \ldots, C_r$. Since $G \in \mathscr{T}_k$, we have $\kappa'(C_0) \leqslant k$. By Menger's Theorem, there is a set E of at most k edges of C_0 such that $C_0 - E$ has exactly two components C_0' and C_0'' joined in C_0 by E. Form $H' = ((H/(\bigcup_{1 \leqslant i \leqslant r} C_i)/C_0')/C_1'$. Then H' has one component of two vertices joined by at most k edges, and its other components are edgeless. Thus H has a nontrivial CH-morph H' with $\bar{\delta}(H') \leqslant k$, so $H \notin \mathscr{S}_k^H$. This contradiction establishes that no nontrivial subgraph of G is in $\mathscr{S}_k^H$, so $G \in (\mathscr{S}_k^H)^R$. Thus $\mathscr{T}_k \subseteq (\mathscr{S}_k^H)^R$, and the equality claimed by the theorem is established. □

Corollary 7.2. *The smallest free graph family containing the k-degenerate graphs is the family of graphs G with $\bar{\kappa}'(G) \leqslant k$.*

Proof. Noting that $\mathscr{S}_k$ is stippled, use Theorems 6.6(b) and 7.1. □

8. Reduction

Let $\mathscr{S}$ be a family of graphs and let G be a graph. We say that G is *$\mathscr{S}$-reduced* if no nontrivial subgraph of G is in $\mathscr{S}$, i.e., if $G \in \mathscr{S}^R$. We say that a graph G_0 is a *$\mathscr{S}$-reduction* of G if G_0 is $\mathscr{S}$-reduced and if G_0 can be obtained from G by a sequence of contractions of subgraphs in $\mathscr{S}$. When the $\mathscr{S}$-reduction is unique, we denote the $\mathscr{S}$-reduction of G by $G/\mathscr{S}$. The subgraphs found in $\mathscr{S}$ that are being contracted may not all be subgraphs of G itself, for some may be created by a prior contraction in the sequence.

The next theorem is given in [5] without the condition of connectedness on $\mathscr{C}$. But, in fact, completeness alone is not sufficient. For example, in Example 3.11 $G/G_1 = 4 * K_2$ while $G/G_2 = 2 * K_2$. Since neither $4 * K_2$ nor $2 * K_2$ has a nontrivial subgraph in $\mathscr{C}$, uniqueness fails. However, the proofs of Theorem 4.1 and Corollary 4.2 given in [5] are valid for connected complete graph families.

Theorem 8.1 ([5] 4.1). *If $\mathscr{C}$ is a connected complete graph family and G is a graph, then the $\mathscr{C}$-reduction of G is unique.*

Theorem 8.2. *Let $\mathscr{C}$ be a connected complete graph family and let G be a graph. Then $E(G/\mathscr{C})$ is the set of edges of G that lie in no subgraph of G that is in $\mathscr{C}$.*

Proof. By Theorem 3.12, $G/\mathscr{C}$ can be obtained from G by contracting each maximal connected subgraph of G in $\mathscr{C}$ to a vertex. (This was the definition of a $\mathscr{C}$-reduction used in [5].) Thus Theorem 8.2 follows by Theorem 3.7. $\square$

Theorem 8.3. *Let $\mathscr{S}$ be a connected family of graphs which is closed under CH-morphisms. Then*

$$(\mathscr{S}^R)^C = (\mathscr{S}^R)^H$$

and for any graph G, the unique $\mathscr{S}$-reduction of G is $G/\mathscr{S} = G/((\mathscr{S}^R)^C)$.

Proof. Since $\mathscr{S}$ is connected and closed under CH-morphisms, by Theorem 4.5, $\mathscr{S}^R$ is a free graph family. Hence, by Lemma 6.4,

$$(\mathscr{S}^R)^C = (\mathscr{S}^R)^H.$$

We must now show the $\mathscr{S}$-reduction is unique. To simplify notation, denote $\mathscr{C} = (\mathscr{S}^R)^C = (\mathscr{S}^R)^H$. By Theorem 6.3, $\mathscr{C} = (\mathscr{S}^R)^C$ is a complete graph family. Since $\mathscr{S}^R$ is free, it is stippled, so by Theorem 3.8, $\mathscr{C}$ is connected. By Theorem 8.1, there is a unique $\mathscr{C}$-reduction of G, denoted $G/\mathscr{C}$.

Let G_0 be a $\mathscr{S}$-reduction of G and let $G_1 = G/\mathscr{C}$. Since G_0 and G_1 are both formed by contracting edges of G, to prove $G_0 = G_1$ it suffices to show

$$E(G_0) = E(G_1).$$

Let $e \in E(G) - E(G_1)$. By Theorem 8.2, since $\mathscr{C}$ is connected and complete, e is in a subgraph $H(e)$ of G lying in $\mathscr{C} = (\mathscr{S}^R)^C$. By Lemma 4.7, $H(e)$ contains a nontrivial subgraph in $\mathscr{S}$. By $\mathscr{C}$ satisfying (C2) and by Lemma 4.7, every nontrivial contraction of $H(e)$ also contains a nontrivial subgraph in $\mathscr{S}$. Therefore, as G is changed to the reduced graph G_0 by a sequence of contractions of members of $\mathscr{S}$, $H(e)$ must be contracted to an edgeless subgraph, since G_0 is reduced. Hence, $e \notin E(G_0)$, and so $E(G_0) \subseteq E(G_1)$.

Let $e \in E(G_1) - E(G_0)$. Then some contraction of G contains a subgraph $H(e)$ in $\mathscr{S}$ such that $e \in E(H(e))$. We may assume that there exist graphs $H(e) = H_1, H_2, \ldots, H_c$, and edge subsets $X_i \subseteq E(H_{i+1})$ $(1 \leqslant i \leqslant c-1)$ such that H_c is a subgraph of G, such that $H_i = H_{i+1}/X_i$, for each $i = 1, 2, \ldots, c-1$, and such that $H_{i+1}[X_i] \in \mathscr{S}$. By Theorem 6.9(a),

$$\mathscr{S} \subseteq (\mathscr{S}^R)^H = (\mathscr{S}^R)^C = \mathscr{C}. \tag{4}$$

By (4), $H_1 = H_2/X_1 = H(e) \in \mathscr{S} \subseteq \mathscr{C}$ and $H_2[X_1] \in \mathscr{S} \subseteq \mathscr{C}$. Since $\mathscr{C}$ is complete, $H_2 \in \mathscr{C}$ follows from (C3). Inductively, $H_3, \ldots, H_c \in \mathscr{C}$. However, Theorem 8.2 asserts that $E(G_1)$ is the set of edges of G lying in no subgraph found in $\mathscr{C}$. Since $e \in E(H(e)) \subseteq H_c$ and $H_c \in \mathscr{C}$, we conclude that $e \notin E(G_1)$, contrary to our supposition. Theorem 8.3 follows. $\square$

In Theorem 8.3, hypothesizing connectedness and closure under contractions alone is not sufficient to get the uniqueness result, as is shown by the following example. Begin with the disjoint union

$$G_1 = 2K_{3,3} \cup K_1,$$

where $\{a, b, c, w, x, y, z\}$ is an independent set in G_1 such that $\{a, b, c\}, \{w\}$, and $\{x, y, z\}$ lie in distinct components of G_1. Let G_2 be the graph obtained from $G_1 + aw$ by setting $b = y$ and $c = z$. Thus, $G_2 - w$ consists of two edge-disjoint copies of $K_{3,3}$, which we denote by H_1 and H_2, where $\{a, b, c\} \subseteq V(H_1)$ and

$$\{b, c\} = \{y, z\} = V(H_1) \cap V(H_2).$$

Let $\mathscr{S}$ be the graph family of all contractions of $K_{3,3}$, including $K_{3,3}$ itself. Thus, $\mathscr{S}$ is connected and closed under contraction, but not under CH-morphisms, because the graph H_0 obtained from a $K_{3,3}$ by identifying two nonadjacent vertices is not in $\mathscr{S}$. Note that G_2/H_1 and G_2/H_2 are not isomorphic. Also, G_2/H_1 and G_2/H_2 are $\mathscr{S}$-reduced (because every nontrivial member of $\mathscr{S}$ is 3-edge-connected, and the only 3-edge-connected subgraph of G_2/H_1 or G_2/H_2 is isomorphic to H_0, which is not in $\mathscr{S}$). Thus, G_2 has two nonisomorphic $\mathscr{S}$-reductions, and $\mathscr{S}$ is a connected graph family closed under contraction.

Also, note that $\mathscr{S} \nsubseteq (\mathscr{S}^R)^H$ when $\mathscr{S}$ is the graph family of contractions of $K_{3,3}$. The reason is that $H_0 \in \mathscr{S}^R$ (because every nontrivial graph in $\mathscr{S}$ is 3-edge-connected, but $H_0 \notin \mathscr{S}$ and no proper subgraph of H_0 is 3-edge-connected), and hence $K_{3,3} \in \mathscr{S} - (\mathscr{S}^R)^H$. Therefore, the hypothesis in Theorem 6.9(a) of closure under CH-morphisms cannot be relaxed to closure under contraction.

We close this section by mentioning two previously published theorems whose proofs used in an essential way the concept expressed more generally in Theorem 8.3. In [3] and in [12], Catlin and Lai proved, respectively:

Theorem 8.4 (Catlin [3]). *If a graph G is at most* 5 *edges short of being* 4*-edge-connected, then either $G \in F_4$ or G has a cut edge or G is contractible to the Petersen graph.*

Theorem 8.5 (Lai [12]). *Let Z_3 denote the cyclic group of order* 3 *and let G be a* 3*-edge-connected chordal graph. Then G is A-connected for every abelian group A with $|A| \geqslant 3$ if and only if neither of the following holds*:

(a) *G has a block which is isomorphic to K_4*;

(b) *G is the union of two induced subgraphs G_1 and G_2 such that G_1 is not Z_3-connected, such that $G_2 \cong K_4$, and such that the intersection of G_1 and G_2 is an edge*.

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, American Elsevier, New York, 1976.
[2] P.A. Catlin, A reduction method to find spanning Eulerian subgraphs, J. Graph Theory 12 (1988) 29–45.
[3] P.A. Catlin, Double cycle covers and the Petersen graph, J. Graph Theory 13 (1989) 465–483.
[4] P.A. Catlin, Supereulerian graphs, a survey, J. Graph Theory 16 (1992) 177–196.
[5] P.A. Catlin, The reduction of graph families closed under contraction, Discrete Math. 160 (1996) 67–80.
[6] P.A. Catlin, J.W. Grossman, A.M. Hobbs, H.-J. Lai, Fractional arboricity strength and principal partitions in graphs and matroids, Discrete Appl. Math. 40 (1992) 285–302.
[7] W.H. Cunningham, Optimal attack and reinforcement of a network, J. Assoc. Comput. Mach. 32 (1985) 549–561.
[8] F. Jaeger, Nowhere-zero flow problems, in: L.W. Beineke, R.J. Wilson (Eds.), Topics in Graph Theory, 3rd Edition, Academic Press, London, 1988, pp. 70–95.
[9] F. Jaeger, N. Linial, C. Payan, M. Tarsi, Group connectivity of graphs — a nonhomogeneous analogue of nowhere-zero flow properties, J. Combin. Theory, Ser. B 56 (1992) 165–182.
[10] H.-J. Lai, Reduction towards collapsibility, in: Y. Alavi, A. Schwenk (Eds.), Graph Theory, Combinatorics, and Applications, Wiley, New York, 1995, pp. 661–670.
[11] H.-J. Lai, Extending a nowhere-zero 4-flow, J. Graph Theory 30 (1999) 277–287.
[12] H.-J. Lai, Group connectivity of 3-edge-connected chordal graphs, Graphs and Combinatorics 16 (2000) 165–176.
[13] H.-J. Lai, H.Y. Lai, Duality in graph families, Discrete Math. 110 (1992) 165–177.
[14] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. 36 (1961) 445–450.
[15] C.St.J.A. Nash-Williams, Decomposition of finite graphs into forests, J. London Math. Soc. 39 (1964) 12.
[16] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954) 80–91.
[17] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. London Math. Soc. 36 (1961) 221–230.